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Models for a paraconsistent set theory

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Abstract

In this paper the existence of natural models for a paraconsistent version of naive set theory is discussed. These stand apart from the previous attempts due to the presence of some non-monotonic ingredients in the comprehension scheme they fulfill. Particularly, it is proved here that allowing the equality relation in formulae defining sets, within an extensional universe, compels the use of non-monotonic operators. By reviewing the preceding attempts, we show how our models can naturally be obtained as fixed points of some functor acting on a suitable category (stressing the use of fixed-point arguments in obtaining such alternative semantics).
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1. Introduction

It is known from Russell's paradox that the first-order axiomatization of the naive set theory of Cantor and Frege is inconsistent in classical logic. More precisely, some peculiar sets '$\{x \mid \varphi\}$' provided by specific φ-instances of the *comprehension scheme*, i.e. '$\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$', lead to triviality if the underlying logic is classical. The most popular of those is the so-called Russell set, $R :=$ '$\{x \mid x \notin x\}$', for by the law of excluded middle one immediately gets '$R \in R \wedge R \notin R$', an apparent *contradiction*. Naive set theory is thus one

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of the simplest examples of an intuitively correct theory leading so readily and desperately to such a contradiction. Of course not so desperately as it might seem to be.

Although it fortunately appeared that these contradictory sets are by no means essential for the foundations of mathematics, certain logicians have expressed a desire that such inconsistent objects be handled and studied within suitable theories, namely *para(in)consistent* ones. After all, *non-well-founded sets* (also called *hypersets*), which are no more indispensable for the foundations of mathematics, have subsequently found interesting applications in modeling circular phenomena, notably in computer science.[1]

There are many examples in mathematics where the introduction of *imaginary/ideal* objects, though giving some advantage to deal with them, has also forced us to give up some basic properties or principles.[2] Obviously, the price to be paid here concerns the logic in which the theory is embedded, and its possible debilitating effects on classical reasoning and mathematical practice.

For a paraconsistent logician, the motivating consideration is that "minor inconsistencies" should not be allowed to lead, as in classical logic, to irrelevant conclusions. With regard to set theory, it is hardly arguable that '$R \in R \wedge R \notin R$' could be considered as a minor inconsistency on the sole (yet actual) fact that a significant amount of mathematics can be developed so unwisely within the naive theory of sets. Certainly it would be more eloquent to exhibit *non-trivial* models of paraconsistent universes of sets which, at least, contain the *cumulative hierarchy* within their classical part.

However expressive such "non-classical" responses to the paradoxes might be, it must be admitted that in general the techniques used to produce alternative semantics are not devoid of mathematical interest, especially those involving a *fixed-point argument*. In order to illustrate this, we are going to present our models as fixed points of some functor '$\mathcal{F}(\cdot)$' acting on a suitable category.

It has already been shown that models for some non-classical set theories can be obtained in that way from solutions to reflexive domain equations '$X \simeq \mathcal{F}(X)$'. Particularly, models for non-well-founded sets were so constructed by computer scientists using techniques of domain theory (see [1,26]). In turn, the most interesting solutions to the consistency problem for *positive comprehension* principles, to which in fact the models we propose are intimately related, were characterized by such reflexive equations as well. These structures, subsequently called *hyperuniverses*, are topological models of a set theory (which strongly contradicts the axiom of foundation) that is based on a general comprehension scheme restricted to those *positive* formulae in which the classical negation does not occur "explicitly".[3] Accordingly, the universal class $V :=$ '$\{x \mid x = x\}$' should be

[1] For a comprehensive introduction to this topic and its applications, we recommend Barwise and Moss' book [8]. The interested reader should also consult Peter Aczel's book [3].

[2] Adding an 'imaginary' number i such that $i^2 = -1$ to the reals forces one to sacrifice the *ordered* field structure, just as adding non-well-founded sets to ZFC interferes with the formulation of inductive definitions, etc.

[3] For precise references on "positive comprehension", the reader should consult [17] or [23], where a brief historical account of the subject is given.

a set, and this is a singular departure from the limitation of size doctrine of ZF and related set theories.[4]

In positive set theory, beside the universal class V, the complement of Russell's class, namely $\overline{R} :=$ '$\{x \mid x \in x\}$', is also a set. The next step is to deal with the Russell set itself, so we make the move in this paper by showing how a *weak negation* can be incorporated into positive comprehension.

To handle such a contradictory set, the idea is as simple as it is naive: since, according to the law of excluded middle, the Russell set does belong and does not belong to itself, we just take this for granted. Therefore one might intuitively think of a paraconsistent set as an ordered pair of collections which cover the universe: the first part collecting those objects which are supposed to belong to it, the second gathering those which are supposed *not* to belong to it, and where it is now agreed that these two parts may have a non-empty intersection. Thus the trick is just to consider membership and non-membership as somewhat independent but symmetrical *positive* properties.

The goal of this paper is mainly to illustrate how and to what extent this very simple idea can be fruitful, showing at the end that there exist *natural models* for a paraconsistent set theory. We shall review the techniques involved in the previous attempts so as to trace and stress the use of fixed-point arguments. Incidentally, we will point out and distinguish two ways of formalizing set theory, namely *abstraction* and *comprehension*. The distinction especially comes out in the manner in which the corresponding universe of sets is modeled.

We have aimed to make the paper as self-contained as possible, so that it be reasonably accessible to a wide range of logicians. In any case, this should make the presentation and the comparison of the diverse approaches easier. As the reader has probably noticed, we have also included a liberal selection of bibliographical references where he may find more complete treatments and pursue some particular areas of interest to him.

To end this introduction, let us indicate how such a membership ambiguity can easily be concocted within a classical context. Consider a universe $\mathcal{U}$ which consists of a collection U of objects together with a topology on it which might materialize some notion of indiscernibility on U. Then, for any $x \in U$ and $S \subseteq U$, define

$$\begin{cases} x \in_{\mathcal{U}} S \leftrightarrow x \in \overline{S}, \\ x \notin_{\mathcal{U}} S \leftrightarrow x \in \overline{U \setminus S}, \end{cases} \qquad \text{where } \overline{(\cdot)} \text{ is the closure operator on } U.^{5}$$

Note that $x \in S \to x \in_{\mathcal{U}} S$, as well as $x \notin S \to x \notin_{\mathcal{U}} S$. Also, $x \in_{\mathcal{U}} U \setminus S \leftrightarrow x \notin_{\mathcal{U}} S$. But now it is allowed that both $x \in_{\mathcal{U}} S$ and $x \notin_{\mathcal{U}} S$, for some $x \in U$ and $S \subseteq U$, and therefore some kind of relative inconsistency may be observed. Note that the non-contradictory subsets of U are nothing but the clopen subsets in $\mathcal{U}$. It should also be stressed that here $\in_{\mathcal{U}}$ and $\notin_{\mathcal{U}}$ are actually not independent. Anyhow, it is hopeless to define a model for a paraconsistent set theory in this way, seeing that $\in_{\mathcal{U}}$ and $\notin_{\mathcal{U}}$ are subsets of $U \times \mathcal{P}(U)$, not of $U \times U$. Nevertheless, we have a situation here in which a paraconsistent set can already be thought

[4] By the way, a comprehensive bibliography on set theories with a universal set can be found at the following address: http://math.boisestate.edu/~holmes/holmes/setbiblio.html.

[5] Throughout the paper we use a small '$\in$' to denote and distinguish the membership relation in the metatheory from the big '$\in$' in the language of models for some set theory.

of as a covering pair of *closed* subsets of the universe and, as we shall see, this is a very suggestive way of looking at paraconsistent sets, indeed.

2. Preliminaries

In this section we give some basic definitions and specify some notations that will be used throughout the paper. We also point out a key distinction in the formalization of set theory: *comprehension* versus *abstraction*.

2.1. Structures for a paraconsistent set theory

For any set M, let $\mathcal{P}_p(M)$ denote the set of ordered pairs of subsets which cover M, i.e. $\mathcal{P}_p(M) := \{(X, Y) \mid X \cup Y = M\}$.

A *structure* $\mathcal{M}$ for a paraconsistent set theory is formally defined by a non-empty set M together with a function $[\cdot]_{\mathcal{M}}$ from M into $\mathcal{P}_p(M)$, called the *extension function*, which applies any $a \in M$ simultaneously to its *positive* extension $[a]^+_{\mathcal{M}}$ and its *negative* extension $[a]^-_{\mathcal{M}}$, i.e.

$$\mathcal{M} :\equiv \langle M\,;\, [\cdot]_{\mathcal{M}}\rangle, \quad \text{where } [\cdot]_{\mathcal{M}} : M \to \mathcal{P}_p(M), \\ [\cdot]_{\mathcal{M}} = \big([\cdot]^+_{\mathcal{M}}, [\cdot]^-_{\mathcal{M}}\big).$$

Note that this conception of a paraconsistent set leads naturally to a related notion of (strong) extensionality which is the following:

$$\text{for any } a, b \text{ in } M, \quad [a]^+_{\mathcal{M}} = [b]^+_{\mathcal{M}} \text{ and } [a]^-_{\mathcal{M}} = [b]^-_{\mathcal{M}} \quad \text{implies } a = b.$$ [6]

In other words, such a structure $\mathcal{M}$ is said to be *strongly extensional* when the extension function $[\cdot]_{\mathcal{M}}$ is injective; then M can be identified with a subset of $\mathcal{P}_p(M)$, namely the range of $[\cdot]_{\mathcal{M}}$.

Now, by setting

$$\begin{cases} a \in_{\mathcal{M}} b & \text{iff} \quad a \in [b]^+_{\mathcal{M}} \\ a \notin_{\mathcal{M}} b & \text{iff} \quad a \in [b]^-_{\mathcal{M}} \end{cases} \quad \text{for any } a, b \in M,$$

such a structure $\mathcal{M}$ is equally defined by means of two binary relations on M:

$$\mathcal{M} :\equiv \langle M\,;\, \in_{\mathcal{M}}, \notin_{\mathcal{M}}\rangle \quad \text{where } \in_{\mathcal{M}} \cup \notin_{\mathcal{M}} = M \times M.$$

This latter view is adopted in [22][7] and [24]. Basically, $\in_{\mathcal{M}}$ and $\notin_{\mathcal{M}}$ can be considered as *weak negation* of each other, since for some $a \in M$, $[a]^+_{\mathcal{M}} \cap [a]^-_{\mathcal{M}}$ is possibly non-empty. Accordingly, '$x \in y$' can be interpreted as being both '`true`' and '`false`' for some x, y in M. To formalize this, we define the *truth function* $\varepsilon_{\mathcal{M}}$ of the membership relation '$\in$'

[6] Here '$=$' is the meta-theoretic *identity*; the interpretation of the *equality* relation within a structure is discussed in Section 4.

[7] Where $\in_{\mathcal{M}}$ and $\notin_{\mathcal{M}}$ are rather denoted by $\in^+_{\mathcal{M}}$ and $\in^-_{\mathcal{M}}$, respectively.

in M as follows:

$$\begin{cases} \mathtt{t} \in \varepsilon_{\mathcal{M}}(a,b) & \text{iff} \quad a \in_{\mathcal{M}} b \\ \mathtt{f} \in \varepsilon_{\mathcal{M}}(a,b) & \text{iff} \quad a \notin_{\mathcal{M}} b \end{cases} \qquad \text{for any } a, b \in M.$$

In this way, $\varepsilon_{\mathcal{M}}(a,b)$ takes exactly one of the following *truth degrees*:

$$0 := \{\mathtt{f}\}, \quad 1 := \{\mathtt{t}\} \quad \text{or} \quad i := \{\mathtt{t}, \mathtt{f}\}.$$

The set of these truth degrees will be denoted by $T := \{0, i, 1\}$, and in these terms, a structure for a paraconsistent set theory appears as

$$\mathcal{M} :\equiv \langle M\,;\, \varepsilon_{\mathcal{M}} \rangle \quad \text{where } \varepsilon_{\mathcal{M}} : M \times M \to T$$

or equivalently, by defining the *extension function* $[\cdot]_{\mathcal{M}}$ in this setting, as

$$\mathcal{M} :\equiv \big\langle M\,;\, [\cdot]_{\mathcal{M}} \big\rangle \quad \text{where } [\cdot]_{\mathcal{M}} : M \to T^M, \\ y \mapsto \big(x \mapsto \varepsilon_{\mathcal{M}}(x,y)\big).$$

Again, such a structure is said to be *strongly extensional* exactly when the extension function $[\cdot]_{\mathcal{M}}$ thus defined is injective, so that M can be identified with the range of $[\cdot]_{\mathcal{M}}$, which is here a subset of T^M.

To sum up, a paraconsistent set can be seen, on the one hand, as an ordered pair of subsets which cover the universe, that is an element of $\mathcal{P}_p(M)$, and on the other hand, as a function on the universe which takes values in T, that is an element of T^M. It will be convenient in the sequel to be able to exchange one view for the other. So, without any abuse, we shall write:

$$\begin{cases} [x]^+_{\mathcal{M}} = [x]^{-1}_{\mathcal{M}}\{1, i\}, \\ [x]^-_{\mathcal{M}} = [x]^{-1}_{\mathcal{M}}\{0, i\}. \end{cases}$$

Remark 1. We could not end this description without mentioning the "dual" route of the alternative to Russell's paradox: the *paracomplete* or *partial* case, where a set is rather materialized by an ordered pair of *disjoint* parts of the universe. Of course, any (strongly extensional) paraconsistent structure $\mathcal{M}$ naturally gives rise to a (strongly extensional) paracomplete structure $\mathcal{M}^*$ on the same universe M, and vice versa, by way of $(\cdot)^* : (A, B) \mapsto (M \backslash B, M \backslash A)$.

$$\text{In other terms:} \quad \begin{cases} a \in_{\mathcal{M}^*} b & \text{iff} \quad \mathit{not}(a \notin_{\mathcal{M}} b) \\ a \notin_{\mathcal{M}^*} b & \text{iff} \quad \mathit{not}(a \in_{\mathcal{M}} b) \end{cases} \qquad \text{for any } a, b \in M.$$

Note that in the three-valued setting, the difference is not even perceptible unless the meaning of i is specified, namely $i := \{\}$ (neither 'true' nor 'false') for the *paracomplete* case, instead of $i := \{\mathtt{t}, \mathtt{f}\}$ (both 'true' and 'false') for the *paracomplete* one. We shall further discuss this apparent duality in Section 3.[8]

[8] More on the duality between the paraconsistent and the paracomplete cases (and especially the natural duality between closed set and open set logics) can be found for instance in Mortensen's book [27].

2.2. Some notations

So far, only the truth degree of the atomic formula '$x \in y$' in a given structure $\mathcal{M}$ has been defined, namely $|a \in b|_{\mathcal{M}} := \varepsilon_{\mathcal{M}}(a, b)$, for any a, b in M. More generally, the *truth degree* of any formula φ interpreted within a given structure $\mathcal{M}$ is denoted by $|\varphi|_{\mathcal{M}}$. Incidentally, whenever we write $|\varphi|_{\mathcal{M}}$, it is always assumed that an assignation has been given to the possible free variables of φ into M so that the truth degree of φ in $\mathcal{M}$ is actually computable. As usual the computation goes inductively as soon as the *truth functions* of the logical connectives and quantifiers involved are defined. Section 3 is intended to give precise definitions of these, here we just content ourselves with specifying some general notations regarding the language of set theory.

In the sequel, $\mathcal{L}$ informally stands for the first-order language of set theory with '$\in$' as unique primitive relational symbol and with all the logical symbols available. When we want to stress the use of further primitive symbols or specify the logical symbols that are allowed in the formulae, we shall put them between '$\langle \cdots \rangle$' and write $\mathcal{L}\langle \cdots \rangle$ to denote the corresponding language. Hopefully the context should make this convention clear. Now, relatively to any given structure $\mathcal{M}$, $\mathcal{L}_M\langle \cdots \rangle$ will be designating the language $\mathcal{L}\langle \cdots \rangle$ extended by constants naming the elements of M. This is nothing but the usual labor-saving device that will allow us to handle valuations conveniently. Incidentally, for any $\varphi \in \mathcal{L}_M\langle \cdots \rangle$, $\varphi(\bar{x})$ is equally used for denoting the formula φ provided its free variables are among $\bar{x} := x_1, \ldots, x_n$, and then, if $\bar{\tau} := \tau_1, \ldots, \tau_n$ is any list of terms (e.g., variables, constants), $\varphi(\bar{x}|\bar{\tau})$, or most often $\varphi(\bar{\tau})$ when no confusion seems possible, will designate the formula obtained from φ by substituting τ_i for each free occurrence of x_i in φ. In order to handle substitutions freely, we shall also conveniently assume that the terms and formulae of the language have been coded in the metatheory, in such a way as to identify the terms or formulae whose writings differ only in the name of their bound variables. For instance, this can be achieved by using Bourbaki's squares in the definition of the language and consider terms and formulae as formal sequences of symbols in the metatheory. This is implicit in the definition of the so-called *term models* in Section 5.

2.3. Comprehension and abstraction

We shall say that a structure as defined in Section 2.1 is a *model* for some set theory if it fulfills some fragment of the comprehension scheme, in the following sense:

Definition 2. Let $\Sigma \subseteq \mathcal{L}$ be any given fragment of the language of set theory. We say that $\mathcal{M}$ fulfills *Comp*$[\Sigma]$ if for each formula $\varphi(x, \bar{y})$ in Σ, and for any list of parameters $\bar{p}$ in M, there exists b in M such that, for any a in M, $|a \in b|_{\mathcal{M}} = |\varphi(a, \bar{p})|_{\mathcal{M}}$.

Such a 'b' is usually denoted, at least in the metatheory, by the *set abstract* '$\{x \mid \varphi(x, \bar{p})\}$', as it clearly depends on φ and the list of parameters $\bar{p}$. But it should be stressed that such a 'b' is not necessarily unique, unless *extensionality* is required.

However, whether or not extensionality holds, one may also consider such a denotational devise in the language of set theory itself by making use of an *abstraction operator*

'$\{\cdot \mid -\}$'. As a characteristic feature, this step enables us to quantify and abstract over variables occurring free in set abstracts. A typical example of a term of the extended language $\mathcal{L}\langle\{\cdot|-\}\rangle$ is '$\{x \mid \{t \mid x \in x\} \in x\}$'.

This subtle yet important distinction has been rarely emphasized in the literature, so that we shall rather speak of *abstraction* scheme instead of *comprehension* to emphasize that set abstracts may already appear as *terms* in the formula φ involved in the corresponding instance of that scheme, i.e.:

Definition 3. Let $\Sigma \subseteq \mathcal{L}\langle\{\cdot|-\}\rangle$ be any given fragment of the language of set theory with an abstraction operator. We say that $\mathcal{M}$ fulfills *Abst*$[\Sigma]$ if for each formula $\varphi(x, \bar{y})$ in Σ, for any list of parameters $\bar{p}$ in M, and for any a in M, we have $|a \in \{x \mid \varphi(x, \bar{p})\}|_{\mathcal{M}} = |\varphi(a, \bar{p})|_{\mathcal{M}}$.

Of course it is tacitly assumed here that a suitable interpretation of the abstraction operator has been defined in M. Specifically, the so-called *term models* described in Section 5.2 will provide us with typical examples of such structures. There we will discuss extensionality together with the use of set abstracts in the object-language; then the difference between comprehension and abstraction should be clear in view of the results we are going to present.

Finally, it is worth noting that both the comprehension and abstraction schemes can be restated as follows: for each formula $\varphi(x, \bar{y})$ in Σ, for any list of parameters $\bar{p}$ in M, the function $x \mapsto |\varphi(x, \bar{p})|_{\mathcal{M}}$ belongs to the range of $[\cdot]_{\mathcal{M}}$. According to Cantor's theorem, the range of $[\cdot]_{\mathcal{M}}$ can never be equal to T^M, so that our investigations might be summarized by the following simple question:

what can the range of $[\cdot]_{\mathcal{M}}$ *be*?

In a sense, this paper is devoted to showing how, by providing T (and M) with some suitable structure, a satisfactory response to this question can be found.

3. The inner logic(s)

The three-valued interpretation described in Section 2.1 leads to revisit the classical meaning of the usual logical operators in that context. It turns out that the deliberate independence of the interpretations in any structure of φ and $\neg\varphi$, for any atomic formula φ, will result in the loss of expressiveness in the system.

As we shall see, this can be offset to some extent by introducing new connectives and thus, depending on the choice of the *primitive* ones (and on the definition of the *consequence* relation), a variety of three-valued *logics*, each having a corresponding variety of merits and defects, comes up. Roughly, it might be said that all the difference lies in the choice of the connective taken as official *implication*. We are not going to specify the primitive connectives here, so that we will not have to refer officially to some logic(s) known in the literature. In any case, a truth-functional characterization of the logical connectives will be largely enough for our investigations.

3.1. A consequence relation

In formal logic, the choice of an appropriate definition of the consequence relation is often guided by a suitable proof-theoretic characterization of the properties of the primitives connectives. For such considerations, the reader is referred to [7]. Here we opted for the common semantical definition.

We first specify the *satisfaction* relation '$\models$' connected with our interpretation:

$$\text{for any structure } \mathcal{M} \text{ and any } \varphi \text{ in } \mathcal{L}_M\langle\cdots\rangle, \quad \mathcal{M} \models \varphi \quad :\text{iff} \quad \mathtt{t} \in |\varphi|_{\mathcal{M}}.$$

Now, for any $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}\langle\cdots\rangle$, define

$$\Sigma \vdash_{\mathtt{t}} \varphi \quad :\text{iff} \quad \text{for any } \widehat{\mathcal{M}} \text{ such that } \widehat{\mathcal{M}} \models \psi, \text{ for any } \psi \in \Sigma, \text{ we have } \widehat{\mathcal{M}} \models \varphi,$$

where $\widehat{\mathcal{M}}$ is denoting a *valuation*, i.e. a structure $\mathcal{M}$ together with a specific assignation in M of the variables occurring free in the formulae of $\Sigma \cup \{\varphi\}$. Thus defined, this consequence relation expresses nothing but the *persistence of* '$\mathtt{true}$' through valuations. Note that in classical logic, the transmission of truth is obviously equivalent to the transmission of *non-falsity*. This is not the case here, so that '$\vdash_{\mathtt{t}}$' actually says nothing about this latter. We will see below how this lack of symmetry interferes with the properties of some connectives in the object language. By the way, it should be said that other semantical variants for the definition of the consequence relation are possible (see [7]).

3.2. Monotonic connectives

From now on we strongly recommend the non-initiate reader to read and understand "$\mathtt{t} \in |\varphi|_{\mathcal{M}}$" as "$\varphi$ *is true in* $\mathcal{M}$" and "$\mathtt{f} \in |\varphi|_{\mathcal{M}}$" as "$\varphi$ *is false in* $\mathcal{M}$". Thereby, a pleasant and natural way of introducing the truth functions of the basic connectives and quantifiers is simply to translate the *classical* rules characterizing them. To distinguish the new logical operators so obtained from their corresponding metatheoretical ones, these latter will be expressed in English.

The truth functions of '$\neg$', '$\wedge$' and '$\forall$' are thus defined by the following rules:

$$\begin{array}{rll}
\mathtt{t} \in |\neg\varphi|_{\mathcal{M}} & \text{iff} & \mathtt{f} \in |\varphi|_{\mathcal{M}}, \\
\mathtt{f} \in |\neg\varphi|_{\mathcal{M}} & \text{iff} & \mathtt{t} \in |\varphi|_{\mathcal{M}}; \\[1ex]
\mathtt{t} \in |\varphi \wedge \psi|_{\mathcal{M}} & \text{iff} & \mathtt{t} \in |\varphi|_{\mathcal{M}} \text{ and } \mathtt{t} \in |\psi|_{\mathcal{M}}, \\
\mathtt{f} \in |\varphi \wedge \psi|_{\mathcal{M}} & \text{iff} & \mathtt{f} \in |\varphi|_{\mathcal{M}} \text{ or } \mathtt{f} \in |\psi|_{\mathcal{M}}; \\[1ex]
\mathtt{t} \in |\forall x\varphi|_{\mathcal{M}} & \text{iff} & \text{for any } a \in M,\ \mathtt{t} \in |\varphi(x|a)|_{\mathcal{M}}, \\
\mathtt{f} \in |\forall x\varphi|_{\mathcal{M}} & \text{iff} & \text{there exists } a \text{ in } M \text{ such that } \mathtt{f} \in |\varphi(x|a)|_{\mathcal{M}}.
\end{array}$$

Those of '$\vee$' and '$\exists$' are obtained in the same way and so remain *classically* definable from these above. This also yields the following tables for the truth functions of the basic connectives, where '$\supset$' stands for the *material* conditional defined by '$\neg\varphi \vee \psi$', and '$\supset\!\subset$'

is the related biconditional connective:[9]

$$
\begin{array}{c|c}
\neg & \\
\hline
1 & 0 \\
i & i \\
0 & 1
\end{array}
\qquad
\begin{array}{c|ccc}
\wedge & 1 & i & 0 \\
\hline
1 & 1 & i & 0 \\
i & i & i & 0 \\
0 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c|ccc}
\vee & 1 & i & 0 \\
\hline
1 & 1 & 1 & 1 \\
i & 1 & i & i \\
0 & 1 & i & 0
\end{array}
\qquad
\begin{array}{c|ccc}
\supset & 1 & i & 0 \\
\hline
1 & 1 & i & 0 \\
i & 1 & i & i \\
0 & 1 & 1 & 1
\end{array}
\qquad
\begin{array}{c|ccc}
\supset\!\subset & 1 & i & 0 \\
\hline
1 & 1 & i & 0 \\
i & i & i & i \\
0 & 0 & i & 1
\end{array}
$$

‘Monotonic connectives.’

It is well known that these truth tables correspond to those of Kleene’s strong logical connectives, where then the middle degree i is rather interpreted as a *truth value gap*, i.e. $i := \{\}$, instead of a *truth value glut*, $i := \{\mathtt{t}, \mathtt{f}\}$. As a characteristic feature i can thus be thought of in both cases as an *imaginary truth value* satisfying $\neg i = i$. In the paraconsistent case, we have $|\varphi|_{\mathcal{M}} = i$ iff $\mathcal{M} \models \varphi \wedge \neg\varphi$, namely φ is *contradictory* in $\mathcal{M}$, while in the paracomplete case, we have $|\varphi|_{\mathcal{M}} = i$ iff $\mathcal{M} \not\models \varphi \vee \neg\varphi$, namely φ is *undetermined* in $\mathcal{M}$. Note that in both cases, the set of truth degrees $T = \{0, i, 1\}$ is naturally ordered by the inclusion relation.[10] These orderings, which are obviously dual of each other, will be referred to in this paper as the *information ordering* $\leqslant_I$ and the *knowledge ordering* $\leqslant_K$, so that in the first case i might be considered as the token of a *clash of information*, and in the second case as the one of a *lack of knowledge*. The ordered sets thus defined are denoted respectively by Λ and V:

$$
\Lambda := \langle T\,; \leqslant_I\rangle \equiv \left\{ \begin{array}{ccc} & i & \\ / & & \backslash \\ 0 & & 1 \end{array} \right. \quad \text{and} \quad V := \langle T\,; \leqslant_K\rangle \equiv \left\{ \begin{array}{ccc} 0 & & 1 \\ \backslash & & / \\ & i & \end{array} \right.
$$

With regard to the consequence relation defined in Section 3.1, a singular departure is that i is a *designated*[11] value in the paraconsistent case (since $\mathtt{t} \in i$) whereas it is not in the paracomplete one. Therefrom it is readily shown that, with the sole connectives defined hitherto, there are no paracomplete tautologies at all, while it can be proved that the paraconsistent tautologies are still exactly the classical ones (see [28] or [7]). Another manifestation of that “asymmetry”, in connection with set theory, is the following.

It is proved in [29] that there exist *non-trivial* paraconsistent models of a full comprehension scheme, namely: there exists $\mathcal{M}$, with $\mathcal{M} \not\models \psi$ for some ψ, such that for any $\varphi(x, \bar{z}) \in \mathcal{L}\langle \in; \neg, \wedge, \forall\rangle$, $\mathcal{M} \models \forall \bar{z}\, \exists y \forall x (x \in y \supset\!\subset \varphi(x, \bar{z}))$. Particularly, in such a model we should have $|R \in R \supset\!\subset R \notin R|_{\mathcal{M}} = i$, for it is readily seen from the ‘$\supset\!\subset$’ table that this can never be equal to 1, which incidentally shows that on the other hand the above scheme cannot be satisfied by any paracomplete structure. By no means should this be interpreted as a defect of the paracomplete interpretation in comparison with the paraconsistent one, because the comprehension scheme as formulated here above does not express (in both cases actually) what it is intended to; in particular it does not fit in with our definition of comprehension given in Section 2.3, seeing that $\mathtt{t} \in |\varphi \supset\!\subset \psi|_{\mathcal{M}}$ does not imply that $|\varphi|_{\mathcal{M}} = |\psi|_{\mathcal{M}}$.

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[9] Note that we are using the same notation for the connectives and their truth functions.

[10] If need were, we would remind the reader that the truth degrees are subsets of $\{\mathtt{t}, \mathtt{f}\}$.

[11] In any many-valued setting, a truth-value/degree v is said to be *designated* if the satisfaction relation ‘$\models$’ is defined in such a way that $\mathcal{M} \models \varphi$ holds whenever $|\varphi|_{\mathcal{M}} = v$.

In fact it is the use of '$\supset$' as official "implication" connective which is irrelevant. Indeed, this connective is not in any sense an *implication* for it is easily shown that *modus ponens* fails, namely $\{\varphi, \varphi \supset \psi\} \nvdash_{\mathtt{t}} \psi$. Though this holds for the paracomplete interpretation, '$\supset$' cannot be considered either as an implication since then $\nvdash_{\mathtt{t}} \varphi \supset \varphi$. In the paraconsistent case, it is however worthy of note that ψ is actually derivable from φ and $\varphi \supset \psi$ as long as φ is non-contradictory. But the problem is, precisely, that such a logic is unable to express that one of its formulae is non-contradictory/determined. For if it was possible to define an unary connective, say '$\circ$', such that $\mathcal{M} \models \circ\varphi$ iff $|\varphi|_{\mathcal{M}} \neq i$, we should have $\circ i = 0$ and $\circ 0 = 1$ or i, and so, in any case, $\circ i \nleqslant_K \circ 0$. Now it suffices to observe that all the connectives considered so far (and so anyone definable from these) are *monotonic* with respect to the knowledge/information ordering.[12] The quantifiers '$\forall$' and '$\exists$' thought of as generalized conjunction and disjunction are easily seen to be monotonic as well. By the way, it will be convenient in the sequel to consider any truth value 0, 1, i as a propositional constant with the corresponding constant truth function, so that these are obviously monotonic too. However, none of these is definable from '$\neg$' and '$\wedge$' alone. But note that in any set theoretical framework, i can be materialized by '$R \in R$'.

This limitative yet remarkable property has been largely explored in the literature and might be considered as a kind of *safety property*[13] with regard to the semantical and logical paradoxes. This will be illustrated and further discussed in Section 5. By way of appetizer, let us just state here that *monotonicity* is actually a guarantee for the existence of a *fixed* value of a truth function on V.[14]

Now the issue we have to face is how to define *in a natural way* new logical connectives that will increase the expressive power of the logic.

3.3. Non-monotonic connectives

Though the introduction of non-monotonic connectives can be carried out concurrently, we shall only focus on the paraconsistent case here. The point is that some significant differences in the truth tables of the corresponding connectives will follow from the schizophrenic interpretation of i, i.e. $i = \{\}$ or $i = \{\mathtt{t}, \mathtt{f}\}$ (whereas there is no difference at all in the tables as far as only monotonic connectives are considered). Again this can be explained by the fact that i is designated in one case and not in the other.

The need of a "true" *implication* connective is apparent from Section 3.2. A natural way of introducing such a one '$\to$' is as follows:

$$\mathtt{t} \in |\varphi \to \psi|_{\mathcal{M}} \quad \text{iff} \quad \mathtt{t} \in |\varphi|_{\mathcal{M}} \text{ implies } \mathtt{t} \in |\psi|_{\mathcal{M}},$$
$$\mathtt{f} \in |\varphi \to \psi|_{\mathcal{M}} \quad \text{iff} \quad \mathtt{t} \in |\varphi|_{\mathcal{M}} \text{ and } \mathtt{f} \in |\psi|_{\mathcal{M}}.$$

[12] Where $f : T^n \to T$ is said to be *monotonic* if $f(x_1, \ldots, x_n) \leqslant_K f(y_1, \ldots, y_n)$ whenever $x_i \leqslant_K y_i$, for $i = 1, \ldots, n$. [NB: this amounts to the same thing when $\leqslant_K$ is replaced by $\leqslant_I$.]

[13] In referring to the title of Avron's evening lecture at the ESSLLI 2002, of which the WoPaLo was a part.

[14] To pursue a previous footnote, adding an imaginary truth value i such that $\neg i = i$ yields a solution to any reflexive *propositional* equation $f(x) = x$, just as adding an imaginary number i such that $i^2 = -1$ to the reals supplies any reflexive *polynomial* equation $f(x) = x$ with a solution.

As this connective is just expressing the *transmission of truth*, it is easily seen that '$\rightarrow$' has the *deduction property* for '$\vdash_{\mathtt{t}}$', stating that it is precisely the translation of the consequence relation at the object-language level:

$$\Sigma \cup \{\varphi\} \vdash_{\mathtt{t}} \psi \quad \text{iff} \quad \Sigma \vdash_{\mathtt{t}} \varphi \rightarrow \psi.$$

In fact, it can be shown that the $\{\wedge, \vee, \rightarrow, \forall, \exists\}$-fragment of $\mathcal{L}$ equipped with '$\vdash_{\mathtt{t}}$' is actually identical to the corresponding fragment of the two-valued classical logic (see [7]). That all the difference retires into the non-classical negation '$\neg$' is not a surprise. By the way, it is worth noticing that '$\neg$' fails to satisfy the substitutivity property (in other words, it is an *intensional* operator):

$$\Phi \dashv\vdash_{\mathtt{t}} \Psi \quad \text{does } \textit{not} \text{ imply} \quad \neg\Phi \dashv\vdash_{\mathtt{t}} \neg\Psi.[15]$$

Of course, this "intensionality" is nothing but the reflect at the object-language level of the apparent and deliberate independence of the interpretations of $\in$ and $\notin$ within any structure. A substitutable *external* negation '$\sim$' can be obtained by setting $\sim\varphi :\equiv \varphi \rightarrow 0$. It is seen from the table of '$\sim$' (below) that the unary connective '$\circ$' defined by $\circ\varphi :\equiv \sim(\varphi \wedge \neg\varphi)$ is then precisely the indicator connective requested in Section 3.2.

As mentioned in Section 3.1, the consequence relation says nothing about the transmission of non-falsity. Consequently, '$\rightarrow$' is not *self-contrapositive*. To remedy this, we may introduce another implication connective defined by:

$$\varphi \Rightarrow \psi :\equiv (\varphi \rightarrow \psi) \wedge (\neg\psi \rightarrow \neg\varphi).$$

It has no longer the deduction property but does still satisfy *modus ponens*. Note that all the conditionals defined so far share a common "classical" negation, i.e.:

$$\mathtt{f} \in |\varphi \supset \psi|_{\mathcal{M}} \quad \text{iff} \quad \mathtt{f} \in |\varphi \rightarrow \psi|_{\mathcal{M}} \quad \text{iff} \quad \mathtt{f} \in |\varphi \Rightarrow \psi|_{\mathcal{M}} \quad \text{iff} \quad \mathtt{t} \in |\varphi|_{\mathcal{M}}$$
$$\text{and } \mathtt{f} \in |\psi|_{\mathcal{M}}.$$

Here are the truth tables of the connectives we have introduced in this section:

$\sim$	
1	0
i	0
0	1

$\circ$	
1	1
i	0
0	1

$\rightarrow$	1	i	0
1	1	i	0
i	1	i	0
0	1	1	1

$\Rightarrow$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

$\Leftrightarrow$	1	i	0
1	1	0	0
i	0	i	0
0	0	0	1

'*Non-monotonic connectives.*'

It then can be confirmed that these are *not* monotonic. As a token of this fact, observe that '$\sim$' has no fixed values. The consequence is that the existence of the set '$\{x \mid \sim(x \in x)\}$' is prohibited. Hence the presence of non-monotonic operators in formulae defining sets can be devastating with regard to set theory.

By the way, a very characteristic property of '$\Leftrightarrow$' is the following:

$$\mathtt{t} \in |\varphi \Leftrightarrow \psi|_{\mathcal{M}} \quad \text{iff} \quad |\varphi|_{\mathcal{M}} = |\psi|_{\mathcal{M}}.$$

[15] Here '$\Phi \dashv\vdash_{\mathtt{t}} \Psi$' means '$\Phi \vdash_{\mathtt{t}} \Psi$ and $\Psi \vdash_{\mathtt{t}} \Phi$', or equivalently '$\vdash_{\mathtt{t}} \Phi \leftrightarrow \Psi$'.

Thus the comprehension/abstraction schemes stated in Section 2.3 can be entirely expressed at the object-language level now:

$$Comp[\Sigma] \equiv \left| \begin{array}{l} \text{for any } \varphi(x, \bar{y}) \text{ in } \Sigma \subseteq \mathcal{L}, \\ \forall \bar{p}\, \exists y\, \forall x (x \in y \Leftrightarrow \varphi(x, \bar{p})). \end{array} \right.$$

$$Abst[\Sigma] \equiv \left| \begin{array}{l} \text{for any } \varphi(x, \bar{y}) \text{ in } \Sigma \subseteq \mathcal{L}\langle\{\cdot\,|-\}\rangle, \\ \forall \bar{p}\, \forall x (x \in \{x \,|\, \varphi(x, \bar{p})\} \Leftrightarrow \varphi(x, \bar{p})). \end{array} \right.$$

Remark 4. Note that by using an abstractor operator, the existential quantifier disappears in the formulation of the abstraction scheme, so that this latter may equally be reformulated by a "double two-ways rule":

$$\begin{cases} x \in \{x \,|\, \varphi(x, \bar{p})\} \dashv\vdash_{\mathtt{t}} \varphi(x, \bar{p}), \\ x \notin \{x \,|\, \varphi(x, \bar{p})\} \dashv\vdash_{\mathtt{t}} \neg\varphi(x, \bar{p}). \end{cases}$$

4. Equality and extensionality

Here it is shown that with the appropriate connectives in their formulations the classical concepts of strict and extensional equalities can be so defined as to maintain their intuitive meaning and classical properties.

4.1. *The equality relation*

In any set theory the role of the equality relation in the language, provided that it can be used in formulae defining sets, is easily demonstrated. Indeed, to define such simple objects as singletons might turn out to be ridiculously cumbersome, and certainly counter-intuitive, if the equality was not available.

In order that the binary relational symbol '$=$' deserves the *equality* status in the present context, we require that its truth function on any structure $\mathcal{M}$, i.e. $=_{\mathcal{M}} : M \times M \to T$, be defined in such a way that the following rules hold:

$$\begin{cases} \vdash_{\mathtt{t}} x = x & & (\textit{reflexivity}) \\ \{x = y, \varphi\} \vdash_{\mathtt{t}} \varphi[x \,|\, y], & \text{for any formula } \varphi^{16} & (\textit{substitutivity}). \end{cases}$$

Note that as soon as the negation can be used we have $x = y \vdash_{\mathtt{t}} \varphi(x) \Leftrightarrow \varphi[x \,|\, y]$, which means that in any structure $\mathcal{M}$, $|\varphi(a)|_{\mathcal{M}} = |\varphi[b|a]|_{\mathcal{M}}$ whenever $\mathtt{t} \in |a = b|_{\mathcal{M}}$.

Remark 5. The reader should easily convince himself that actually the substitutivity property for $\mathcal{L}$-formulae[17] can be met by only prescribing '$=$' to satisfy $x = y \vdash_{\mathtt{t}} x \underset{\circ}{=} y \wedge x \overset{\circ}{=} y$,

[16] Where $\varphi[x \,|\, y]$ is denoting the formula obtained from φ by substituting y for some, but not necessarily all, free occurrences of x.

[17] So in which '$=$' does not occur.

where the following abbreviations are used:

$$x \underset{\circ}{=} y :\equiv \forall z(z \in x \Leftrightarrow z \in y) \quad \text{and} \quad x \stackrel{\circ}{=} y :\equiv \forall z(x \in z \Leftrightarrow y \in z).$$

To single out a particular class of structures that play a central role, let us now adopt the next terminology:

Definition 6. We say that a structure $\mathcal{M}$ is *normal* if the equality relation is interpreted in such a way that, for any $a, b \in M$, $\mathtt{t} \in |a = b|_{\mathcal{M}}$ iff $a = b$ in M.[18]

It should be remarked that this does not completely determine the truth function of '$=$' on M. All that is implied in the paraconsistent case is

$$\begin{cases} |a = b|_{\mathcal{M}} = 1 \ \underline{\text{or}}\ i & \text{iff} \quad a = b \text{ in } M, \\ |a = b|_{\mathcal{M}} = 0 & \text{iff} \quad a \neq b \text{ in } M. \end{cases}$$

Thus, the same object could be interpreted within a normal structure as being both equal to and different from itself. If it is the case that $|a = a|_{\mathcal{M}} = 1$ for any a in M, then '$=_{\mathcal{M}}$' is called the *classical identity* on M.

4.2. About extensionality

The *extensionality principle* states that "two sets are equal as soon as it is shown that they share the same members". Accordingly, we shall say that a structure is *extensional* when it obeys this principle. The relation of "sharing the same members" can be interpreted in any structure $\mathcal{M}$ by "having the same extension(s)", which is exactly expressed by '$\underset{\circ}{=}$', so this latter is often referred to as the *extensional equality*. The extensionality principle does precisely assert that this latter is indeed an *equality* in the sense of Section 4.1. Now, depending on whether the equality relation is admitted in the language, a structure $\mathcal{M}$ can be said to be extensional in two ways essentially:

- If the use of '$=$' is stipulated, the extensionality principle should appear as the rule $x \underset{\circ}{=} y \vdash_{\mathtt{t}} x = y$ (and so we would have $\vdash_{\mathtt{t}} x \underset{\circ}{=} y \leftrightarrow x = y$). Then we say that $\mathcal{M}$ is *extensional* if $\mathcal{M} \models \forall x \forall y(x \underset{\circ}{=} y \to x = y)$.
- If the use of '$=$' is proscribed, according to the remark in Section 4.1, '$\underset{\circ}{=}$' could be considered as an equality provided that the rule $x \underset{\circ}{=} y \vdash_{\mathtt{t}} x \stackrel{\circ}{=} y$ holds. Then we say that $\mathcal{M}$ is *weakly extensional* if $\mathcal{M} \models \forall x \forall y(x \underset{\circ}{=} y \to x \stackrel{\circ}{=} y)$.

Remark 7. If for any $a \in M$, there is $\breve{a} \in M$ with $\mathcal{M} \models \forall x(x \in \breve{a} \Leftrightarrow a \in x)$, i.e. $\breve{a} =$ '$\{x \mid a \in x\}$', then it is easily proved that $\mathcal{M} \models \forall x \forall y(x \stackrel{\circ}{=} y \Rightarrow x \underset{\circ}{=} y)$. This should be the case in any model for some version of naive set theory, showing that '$\underset{\circ}{=}$' and '$\stackrel{\circ}{=}$' are intimately related to each other, independently of extensionality.

[18] This latter equality symbol is the meta-theoretic *identity*; explicitly, '$a = b$ in M' means that 'a' and 'b' are designating the same object in M.

Any extensional structure is obviously weakly extensional. Conversely, it can be seen that any weakly extensional structure $\mathcal{M}$ gives rise to an extensional one by defining $=_{\mathcal{M}}$ on M by $|x = y|_{\mathcal{M}} := |x \underset{\circ}{=} y|_{\mathcal{M}}$. To see this, one has to prove that this interpretation of '$=$' fulfills reflexivity and substitutivity in Section 4.1.

Hint. Remark that $x \underset{\circ}{\neq} y \dashv\vdash_{\mathtt{t}} \delta(x, y)$, where this latter is the $\mathcal{L}$-formula defined by $\delta(x, y) :\equiv \exists z((z \in x \wedge z \notin y) \vee (z \notin x \wedge z \in y))$.

As in any case the extensionality principle asserts that $x = y \dashv\vdash_{\mathtt{t}} x \underset{\circ}{=} y$, it is interesting to note that within an extensional universe, by allowing '$=$' in formulae defining sets, some non-monotonic functions may sneak into them by the back door.[19] This is going to have to be explained in the last sections.

Finally, it should be recalled that, previously in Section 2.1, a structure $\mathcal{M}$ was said to be *strongly extensional* if for any a, b in M, $\mathcal{M} \models a \underset{\circ}{=} b$ implies $a = b$ in M.[19] In fact this now coincides with the notion of *extensional normal* structure, i.e.:

$$\textit{strongly extensional} \equiv \textit{extensional} + \textit{normal}.$$

It is worth noting that in any extensional normal structure $\mathcal{M}$, for any distinct objects a, b in M, as $\mathtt{t} \notin |a = b|_{\mathcal{M}}$, we must have $\mathtt{t} \notin |a \underset{\circ}{=} b|_{\mathcal{M}}$, and so $\mathtt{f} \in |a \underset{\circ}{=} b|_{\mathcal{M}}$. Therefore, if moreover, for any a in M such that $\mathtt{f} \in |a = a|_{\mathcal{M}}$, it was the case that $\mathtt{f} \in |a \underset{\circ}{=} a|_{\mathcal{M}}$ as well, then we would have that $\mathcal{M} \models \forall x \forall y(x \underset{\circ}{=} y \Rightarrow x = y)$. Such a strongly extensional structure is said to be *perfect*. The models which are described in Section 5 fulfill this stronger axiom of extensionality, considering that '$=$' will be so defined as to satisfy $x \neq y \dashv\vdash_{\mathtt{t}} \delta(x, y)$ (with $\delta(x, y)$ as above). Consequently, within such models the contradictory sets are exactly those that are different from themselves, namely $x \neq x \equiv \delta(x, x) \equiv$ "*x is contradictory*".

5. Kripke-style models: a tribute to monotonicity

Historically, the first "successful" attempts of building a universe of sets which is governed by some kind of full comprehension scheme in a non-classical logic were based on a *term model* construction (e.g., [9,20]). Roughly, the universe M of such a model is simply made of *set abstracts*, seen as *syntactical* expressions of the form $\{x \mid \varphi(x)\}$, say for suitable formulae φ, and then, by a *fixed-point argument*, the membership relation on M is determined in such a way that finally $\{x \mid \varphi(x)\}$ be a solution to the corresponding instance of what is now called the *abstraction scheme*.

Basically, that technique initiated by Gilmore in [19], and then resumed by Brady in [9], both for the paracomplete case, can be considered as the counterpart of the seminal but later work by Kripke on the liar paradox. This parallel has already been noticed and largely investigated by Feferman in [16]. For a comprehensive treatment concerning the liar paradox only, the reader is referred to Visser's paper [30].

[19] More precisely, by means of the '$\Leftrightarrow$' involved in the definition of '$\underset{\circ}{=}$'.

In this section, we shall further explore the subject to point out the main defect of such attempts, namely the incompatibility of extensionality and abstraction with equality in the language.

5.1. A fixed-point theorem

We start with showing how/why a fixed-point argument comes into the picture.

Let $f(p)$ be any propositional function in one propositional variable p, and let $\tau :=$ '$\{x|\ f(x \in x)\}$'. If the existence of τ was guaranteed in a structure $\mathcal{M}$, we would have $|\tau \in \tau|_{\mathcal{M}} = |f(\tau \in \tau)|_{\mathcal{M}} = |f|(|\tau \in \tau|_{\mathcal{M}})$, where $|f|$ is denoting the truth function of f, showing that this latter should have a fixed-point. Thus, roughly, sets defined by means of truth functions having fixed-points are most welcome. As furtively mentioned in Section 3.2, *monotonic* truth functions on V/Λ are such ones; in that sense monotonicity is a *safety* property.

In order-theoretic terms, stating that any monotonic function on V/Λ has a fixed-point amounts to saying that V is a *dcpo*:

Definition 8. We say that a partially ordered set D is *directed complete*, or in brief is a *dcpo*, if any directed subset in D has a least upper bound; where $A \subseteq D$ is said to be *directed* if for any $a, b \in A$, there exists $c \in A$ with $a, b \leqslant_D c$. [Note that $\emptyset$ is directed, so that D must have a least element (provided $D \neq \emptyset$).]

The relevance of this notion lies in the following important result:

Theorem 9 (Knaster–Tarski). *Let D be a dcpo and $f : D \to D$ be monotonic. Then f has a fixed point, i.e. there exists x in D such $f(x) = x$. Moreover, if Fix(f) denotes the set of its fixed points, then Fix(f) is also a dcpo.*

Using the machinery of ordinal numbers, the least fixed point can be reached inductively by iterating f from the least element of D, while the existence of maximal ones relies on Zorn's lemma (unless D is finite, of course).

The original version of this statement due to Tarski was rather concerned with complete lattices. This one involving dcpo's is actually the best possible refinement in view of the following result (due to Markowski), of which the proof is by far much harder:

> *If D is an ordered set with a least element such that any monotonic $f : D \to D$ has a fixed point, then D is a dcpo.*

The Knaster–Tarski fixed-point theorem, as many other fixed-point theorems actually, has shown itself to be a tremendous tool in many areas for solving so-called *reflexive equations* '$X \simeq \mathcal{F}(X)$'. In a sense, any universe of sets might be conceived as a solution to such a reflexive equation.

The framework of dcpo's rather suggests considering T with the *knowledge ordering* $\leqslant_K$, even in the paraconsistent case, seeing that Λ is not properly speaking a dcpo. As anyway V and Λ have the same class of monotonic functions, as long as only monotonic operators are involved in formulae defining sets, a result for the paracomplete case will yield a corresponding result for the paraconsistent one and vice versa. The introduction of non-monotonic operators (as the equality relation) may introduce a certain asymmetry in the duality. The attitude that consists in considering T with the knowledge ordering when $i = \{\mathtt{t}, \mathtt{f}\}$ can be summarized by the following slogan:

"*a clash of information* $\equiv$ *a lack of knowledge*".

It is time now to proceed to some applications of this fixed-point theorem in connection with set theory.

5.2. A denotational universe

Let Σ be any given fragment of $\mathcal{L}\langle\{\cdot|-\}\rangle$ defined by means of *monotonic* connectives and quantifiers in such a way that Σ be closed for substitutions, i.e. whenever $\varphi(\bar{x})$ is in Σ and $\bar{\tau}$ are Σ-terms, that are variables, Σ-constants or set abstracts $\{x \mid \psi\}$ where ψ is in Σ, then $\varphi(\bar{\tau})$ is also in Σ. In fact, this assumption guarantees that the abstraction operator is naturally interpretable on the set of all *closed* Σ-terms, which is denoted here by M: the interpretation of '$\{x \mid \varphi(x, \bar{y})\}$', for $\bar{y} := \bar{\tau}$ in M, being nothing but $\{x \mid \varphi(x, \bar{\tau})\}$.

We are going to show that among all the structures having M as universe there do exist such ones fulfilling *Abst*$[\Sigma]$. Since the universe is predetermined, we may identify any structure $\mathcal{M}$ with its membership truth function $\varepsilon_{\mathcal{M}} \in V^{M\times M}$. Now the "approximating" function $(\cdot)^+ : V^{M\times M} \to V^{M\times M}$ is defined by:

$$|a \in b|_{(\mathcal{M})^+} := \big|\varphi(x|a)\big|_{\mathcal{M}} \quad \text{for any } a, b \text{ in } M \text{ with } b = \{x \mid \varphi(x)\}.$$

It is clear that the models of *Abst*$[\Sigma]$ are thus exactly the fixed points of $(\cdot)^+$. It then remains to show that such fixed points exist. To do that, observe that $V^{M\times M}$ is itself naturally equipped with a *knowledge/information* ordering, defined by $\mathcal{M}_1 \leqslant_K \mathcal{M}_2$:iff for all x, y in M, $\varepsilon_{\mathcal{M}_1}(x, y) \leqslant_K \varepsilon_{\mathcal{M}_2}(x, y)$. Clearly this ordering turns $V^{M\times M}$ into a dcpo as well. Now, as long as only monotonic connectives and quantifiers are involved in Σ, it can readily be proved that the approximating function $(\cdot)^+$ is monotonic on $V^{M\times M}$. Therefore the existence of models of *Abst*$[\Sigma]$ is established by the fixed-point theorem in Section 5.1.

It is apparent from the meaning of $\leqslant_K$ on $V^{M\times M}$ that the most interesting models should be the *maximal* ones, namely those in which there is a *minimum* of *undetermined/contradictory* "memberships". As a general rule, these models fail to obey the extensionality principle.

With regard to extensionality, a much more problematic aspect on which we shall now elaborate is the use of the equality relation in formulae defining sets.

Note that it is actually possible to incorporate the equality relation, by defining $=_{\mathcal{M}}$ in any structure $\mathcal{M}$ to be the *classical identity* on M, namely:

$$|a = b|_{\mathcal{M}} := \begin{cases} 1 & \text{if } a \text{ and } b \text{ are } \textit{syntactically} \text{ equal}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, by setting $|a = b|_{(\mathcal{M})^+} := |a = b|_{\mathcal{M}}$, the function $(\cdot)^+$ remains monotonic, and then the consistency of an abstraction scheme with equality in formulae defining sets follows:

Theorem 10 (Gilmore). *Abst*$[\mathcal{L}\langle\in, =, \{\cdot|-\}; 0, \neg, \wedge, \forall\rangle]$ *is consistent.*

This result emerges from Gilmore's work on *partial set theory* (see [19,20]). The original proof makes use of partial predicates and it is based on the so-called *inductive method*, later popularized by Kripke, so that the model considered is actually the *minimal* one. It is not extensional, seeing that the syntactical interpretation of the equality is by no means suitable for the intended meaning of the elements of M, as for instance it leads to differentiate $\{x \,|\, 0 \wedge 0\}$ from $\{x \,|\, 0\}$, whereas these are clearly extensionally equal. Accordingly, there would not be any extensional structure.

One might be tempted to remedy the situation by adopting a more suitable definition of the equality in the model itself. That is probably why, in a first report on his result (see [19]), Gilmore suggested that his model might be extensional. He however had to disprove it later in his paper [20]. Indeed, it is hopeless to combine extensionality and abstraction with the equality relation in formulae defining sets. To stress once more the asymmetry (due here to the presence of the equality), it should be noted that the original proof of this result, further explored and slightly simplified by Hinnion in [21], does not apply directly to the paraconsistent case. In fact, and curiously enough, it is less sophisticated for this latter:

Theorem 11. *There is no extensional paraconsistent model of Abst*$[\mathcal{L}\langle\in, =, \{\cdot|-\}; 0\rangle]$.

Proof. Set $a(x) := \{u \mid x \in x\}$ and $\tau := \{x \mid a(x) = \emptyset\}$, where $\emptyset := \{u \mid 0\}$.

Now let $\mathcal{M}$ be any paraconsistent model of *Abst*$[\mathcal{L}\langle\in, =, \{\cdot|-\}; 0\rangle]$.

We show that $\mathcal{M} \models a(\tau) \underset{\circ}{=} \emptyset$, while $\mathcal{M} \not\models a(\tau) = \emptyset$.

To show that $\mathcal{M} \models a(\tau) \underset{\circ}{=} \emptyset$, suppose $\mathtt{t}_\in |u \in a(\tau)|_{\mathcal{M}}$. Hence $\mathtt{t}_\in |\tau \in \tau|_{\mathcal{M}}$, and thus $\mathtt{t}_\in |a(\tau) = \emptyset|_{\mathcal{M}}$. Then, by substitutivity, we would have $\mathtt{t}_\in |u \in \emptyset|_{\mathcal{M}}$, which is impossible. Hence, for any u, $|u \in a(\tau)|_{\mathcal{M}} = 0 = |u \in \emptyset|_{\mathcal{M}}$, as desired.

On the other hand, we must have $|a(\tau) = \emptyset|_{\mathcal{M}} = 0$; for if $\mathtt{t}_\in |a(\tau) = \emptyset|_{\mathcal{M}}$, we would have $\mathtt{t}_\in |\tau \in \tau|_{\mathcal{M}}$, by definition of τ, and so, for any u, $\mathtt{t}_\in |u \in a(\tau)|_{\mathcal{M}}$, which is impossible as we have just seen. $\square$

This observation is all the more interesting since Brady in [9] proved a complementary result of Gilmore's, even though he was not aware of that, namely that one can recover extensionality if (and only if) one drops equality out of the language:[20]

Theorem 12 (Brady). *Abst*$[\mathcal{L}\langle\in, \{\cdot|-\}; 0, \neg, \wedge, \forall\rangle]$ *is consistent with extensionality.*

Brady's ingenious proof consists in showing that the minimal model is weakly extensional (see [9] for the paracomplete case and [12] for the paraconsistent one). It is actually based on the fact that the minimal fixed point of a monotonic function on a dcpo can be reached *inductively*, so that extensionality can indeed be considered as a characteristic

[20] It should be remarked that Brady in [9] (submitted in 1969) could not have been aware of a result of Gilmore's [20] published in 1974, but even in subsequent works [10–12] this fact does not seem to have been noticed.

property of the *minimal* fixed-point model. Incidentally, note that by its nature, the axiom of extensionality is *not* monotonic since the non-monotonic connective '$\rightarrow$' is involved in its formulation. To testify to this fact, it is fairly easy to exhibit *maximal* fixed point models which are *not* extensional:

Proof. Set $\tau := \{x \mid x \in x\}$ and $\tau' := \{x \mid x \in x \wedge x \in x\}$. It is clear that, for any $\mathcal{M}$, $\mathcal{M} \models \tau \mathrel{\mathring{=}} \tau'$. It is easy to see as well that if $\mathcal{M}_0$ is the minimal model, then we must have $|\tau \in \tau|_{\mathcal{M}_0} = i = |\tau' \in \tau'|_{\mathcal{M}_0}$.

We now define a new structure $\mathcal{M}_1$ as follows:

$$\begin{cases} |\tau \in \tau|_{\mathcal{M}_1} = 0 \text{ and } |\tau' \in \tau'|_{\mathcal{M}_1} = 1, \\ |a \in b|_{\mathcal{M}_1} = |a \in b|_{\mathcal{M}_0} \quad \text{otherwise.} \end{cases}$$

Fact. $\mathcal{M}_1 \leqslant_K \mathcal{M}_1^+$.

To see this, first observe that $|\tau \in \tau|_{(\mathcal{M}_1)^+} = |\tau \in \tau|_{\mathcal{M}_1} = 0$ and that $|\tau' \in \tau'|_{(\mathcal{M}_1)^+} = |\tau' \in \tau' \wedge \tau' \in \tau'|_{\mathcal{M}_1} = |\tau' \in \tau'|_{\mathcal{M}_1} \wedge |\tau' \in \tau'|_{\mathcal{M}_1} = 1$.

Now, let a and $b = \{x \mid \varphi(x)\}$ in M with a and b not both equal to τ or τ'. Then, $|a \in b|_{\mathcal{M}_1} = |a \in b|_{\mathcal{M}_0} = |\varphi(x|a)|_{\mathcal{M}_0} \leqslant_K |\varphi(x|a)|_{\mathcal{M}_1} = |a \in b|_{(\mathcal{M}_1)^+}$, seeing that $\mathcal{M}_0 <_K \mathcal{M}_1$ and that φ is supposed to be monotonic.

Note that if $\mathcal{M}_1^*$ was any maximal $(\cdot)^+$-fixed point above $\mathcal{M}_1$, we would have $|\tau \in \tau|_{\mathcal{M}_1^*} = 0$ and $|\tau' \in \tau|_{\mathcal{M}_1^*} = |\tau' \in \tau'|_{\mathcal{M}_1^*} = 1$, and so $\mathcal{M}_1^* \not\models \tau \mathrel{\mathring{=}} \tau'$. Thus $\mathcal{M}_1^*$ could not be extensional. [Note that this already happens in the least fixed-point above $\mathcal{M}_1$.]

To show that such a maximal fixed point $\mathcal{M}_1^*$ does exist is routine:

Let $D_1 := \{\mathcal{M} \mid \mathcal{M}_1 \leqslant_K \mathcal{M}\}$ and $(\cdot)_1^+$ be the restriction of $(\cdot)^+$ to D_1. We first observe that $(\cdot)_1^+$ is monotonic on D_1. Indeed, if $\mathcal{M}_1 \leqslant_K \mathcal{M}$, then we have $\mathcal{M}_1 \leqslant_K \mathcal{M}_1^+ \leqslant_K \mathcal{M}^+$ (using the fact), and this latter belongs to D_1 as well. Therefore, as D_1 is a dcpo, so is $\mathit{Fix}[(\cdot)_1^+]$, and any maximal element suits. $\square$

Remark 13. The inductive method was further investigated by Brady, in [10] for the paraconsistent case and in [11] for the paracomplete one. Roughly, by subtly defining a *non*-truth-functional conditional '$\rightsquigarrow$', he succeeded in proving the consistency of a full abstraction scheme together with a corresponding weak form of extensionality, namely:

$$\mathit{Abst}' :\equiv \left| \begin{array}{l} \text{for any } \varphi(x, \bar{y}) \text{ in } \mathcal{L}\langle \in, \{\cdot \mid -\}; 0, \neg, \wedge, \forall, \rightsquigarrow \rangle, \\ \forall \bar{p}\, \forall x (x \in \{x \mid \varphi(x, \bar{p})\} \leftrightsquigarrow \varphi(x, \bar{p})), \end{array} \right.$$

$$\mathit{Ext}' :\equiv \left| \forall x \forall y\, (\forall z (z \in x \leftrightsquigarrow z \in y) \rightarrow \forall z (x \in z \leftrightsquigarrow y \in z)). \right.$$

In the paracomplete case, it is shown in [11] that the underlying logic is close to the infinite-valued Łukiasiewicz logic, for which besides it was proved that the full abstraction scheme (without equality in the language either) is consistent. Come to this, it is worth mentioning that a similar technique to Brady's has very recently been applied to the liar paradox by Field in [15].[21]

[21] It would seem that Brady has himself applied his technique to the Liar in his book *Universal Logic*, CSLI publs, forthcoming (2003).

5.3. The issue

In view of the results described above, it is rather apparent that such *denotational* universes cannot be considered as “natural models” for some set theory. Actually, the above considerations raise the following question: by dropping set abstracts out of the language but keeping equality, can we get a theory with extensional models? In other words,

is $Comp[\mathcal{L}\langle\in, =; 0, \neg, \wedge, \forall\rangle]$ *consistent with extensionality*?

Yes, we can give a positive answer to this question for the paraconsistent case. The models we are going to present were first introduced and manufactured in a different way in [22], and then further studied in [24]. It seems quite clear that the construction of such models should involve other techniques. The remainder of this paper is devoted to showing how one could have got these in a natural way.

It has already been argued by some authors (e.g., [16]) that the sole use of monotonic connectives and quantifiers in formulae defining sets is not sufficient to be able to develop classical mathematics from a set theoretical point of view. To contrast with this, it should be said the models we propose actually fulfill an extended comprehension scheme involving some non-monotonic functions (due to the presence of ‘$=$’) and then it turns out that the natural paraconsistent set theory to which the properties of these models give rise can interpret ZF.

6. Scott-style models: beyond monotonicity

The second technique of building models which is tackled here has its source in Scott’s work on models for the untyped lambda calculus. Roughly, it was there the discovery that the fixed-point theorem stated in the previous section is actually reflected within suitable subcategories of the corresponding objects, namely the *dcpo’s*, with the appropriate morphisms, the so-called *Scott-continuous* functions, which are specific monotonic functions having a computational meaning. Indeed, it has been proved there is a wide variety of functors ‘$\mathcal{F}(\cdot)$’ which do have fixed points within those categories, such fixed points being obtained by *projective limits*. Since then, this technique has proved to be very useful in theoretical computer science for producing semantics of programming languages.[22]

6.1. The limit of monotonicity

For our purposes it might therefore be tempting to solve a reflexive domain equation of the form $X \simeq \mathcal{F}(X)$ where $\mathcal{F}(X) \subseteq \langle X \to V\rangle$, the set of all monotonic functions from X into V, for it is rather clear that such a fixed point could give rise to a *strongly extensional* model of $Comp[\mathcal{L}\langle\cdots; 0, \neg, \wedge, \forall\rangle]$.

Such an attempt for the paracomplete case is taken very seriously in [5], where it is finally shown that the structure so constructed contains a model of “rough set theory”

[22] We again rely on appropriate references and leave the interested reader to consult the abundant literature on the subject, as [2] for instance.

within its maximal elements (see also [6]). However attractive this idea is, it cannot solve our question, at least as far as monotonicity alone is concerned. To see this, we first have to point out a significant difference between this technique and the one described in the previous section: such a fixed-point model $M \simeq \mathcal{F}(M)$ would be obtained by projective limit within a subcategory of dcpo's, so that the universe M of the final structure, which is not fixed in advance here, carries its own *knowledge/information* ordering $\leqslant_M$ that turns it into a dcpo. Now, whatever the way $\varepsilon_{\mathcal{M}}$ is actually defined on M, the existence of the sets '$\{x \mid x \in x\}$', '$\{x \mid x \in a\}$' and '$\{x \mid a \in x\}$', for any a in M, immediately compels the function $x \mapsto \varepsilon_{\mathcal{M}}(x, x)$ and, for any a in M, the functions $x \mapsto \varepsilon_{\mathcal{M}}(x, a)$ and $x \mapsto \varepsilon_{\mathcal{M}}(a, x)$, to belong to $\mathcal{F}(M) \subseteq \langle M \to V \rangle$. Accordingly, $\varepsilon_{\mathcal{M}} : M \times M \to V$ should be monotonic in both of its arguments with respect to $\leqslant_M$. Of course, the same remark applies to $=_{\mathcal{M}} : M \times M \to V$.

So, in order to be able to incorporate the equality relation in formulae defining sets, this latter should be monotonic. This is unfortunately not the case on a normal model, for as soon as there exist two objects $a, b \in M$ with $a <_M b$, we would have $\mathtt{t} \notin |a = b|_{\mathcal{M}}$ whereas $\mathtt{t} \in |a = a|_{\mathcal{M}}$.

We shall thus abandon the category of dcpo's for another one in which the identity is welcomed. It has often been shown in mathematics that a problem (an equation in particular) can be solved by enlarging the realm of its potential solutions. What we are going to do hereafter is another illustration of this.

6.2. From monotonicity to continuity

It is well known that monotonicity is a particular case of *continuity*. Indeed, if any ordered set is equipped with the topology of which the closed subsets are simply the downwards-closed subsets, i.e. A is *closed* iff $x \leqslant a \in A$ implies $x \in A$, then the monotonic maps become precisely the continuous ones.

Applying this to $V \equiv \langle T; \leqslant_K \rangle$, it is easy to see that

$$f : X \to V \text{ is continuous} \quad \text{iff} \quad f^{-1}\{0, i\} \text{ and } f^{-1}\{1, i\} \text{ are both closed.}$$

Now we would remind the reader that such a function f defines a paraconsistent set x for which precisely $[x]^+_{\mathcal{X}} = f^{-1}\{1, i\}$ and $[x]^-_{\mathcal{X}} = f^{-1}\{0, i\}$. So the next step is naturally to look at paraconsistent sets defined in terms of *closed* subsets which cover the universe, but closed in a broader topological sense, not only in connection with any given ordering. This leads to swap the set of monotonic functions for the set of all continuous functions from a topological space X into V, which then becomes identified with:

$$\mathcal{F}(X) := \big\{(A, B) \mid A, B \text{ closed in } X \text{ and } A \cup B = X\big\} \subseteq \mathcal{P}_p(X).$$

As we shall see, a solution to $X \simeq \mathcal{F}(X)$ will come up to our expectations. To solve this reflexive equation, we shall work within a suitable category of *Hausdorff* topological spaces. This is a singular departure from the preceding attempts because the topology of an ordered set is *not* Hausdorff (unless the ordering is totally disconnected). Now, in spite of the fact that the equality $=_{\mathcal{M}} : M \times M \to V$ is not monotonic on a normal model $\mathcal{M}$, we will see that it can however be continuous, but then $=^{-1}_{\mathcal{M}} \{1, i\} = \{(a, a) \mid a \in M\}$ should be closed, which exactly means that M should be Hausdorff! So we are on the right track.

6.3. Another fixed-point theorem to the rescue

In the light of Section 6.2, the fixed-point theorem in Section 5.1 simply states that particular continuous functions, namely monotonic ones, do have fixed points. Now there are other well-known fixed-point theorems involving continuous maps.

One of the most famous (and much harder to prove by far) is Brouwer's, stating that any real continuous function on the cube $[0,1]^n$ $(n \geqslant 1)$ has a fixed point. Interestingly, this theorem was invoked in the first attempts of building a model for naive set theory in the infinite-valued Łukasiewicz logic.[23]

Another famous fixed-point theorem is Banach's for *complete metric spaces*:

Definition 14. Let $\langle X_1, d_1\rangle, \langle X_2, d_2\rangle$ be metric spaces. We say that $f : X_1 \to X_2$ is *contracting* if there exists $\varepsilon < 1$ such that $d_2(f(x), f(y)) \leqslant \varepsilon \cdot d_1(x, y)$, for any x, y in M.

Theorem 15 (Banach). *Let $\langle X, d\rangle$ be a complete metric space and $f : X \to X$ a contracting function. Then f has a unique fixed point, i.e. there is a unique x in X such that $f(x) = x$.*

Just as for dcpo's, it has been shown that this fixed-point theorem is reflected within certain categories of corresponding objects, the *complete metric spaces*. Indeed, it is proved in [4] that there is a large class of functors, called "*contracting*", that do have fixed points within such a suitable category. It is then not surprising that the framework of complete metric spaces too has shown itself to be a useful tool for producing semantics of programming languages.

6.4. Working with compact complete metric spaces

Roughly, using the techniques described in [4], a solution to our reflexive equation $X \simeq \mathcal{F}(X)$ can be found. What we get then is a *compact* metric space M together with an *homeomorphism* h from M onto $\mathcal{F}(M)$. As we shall see, *compactness* is going to be an essential ingredient in order to show that such a structure gives rise to a model for some paraconsistent set theory. Of course it is beyond the scope of this paper to describe the construction of that structure. It will not be necessary anyway. As previously mentioned, such a solution is already presented and manufactured in a different way in [22]. Moreover, it should be remarked that the results of [4] cannot be applied directly to our case. Some technical and precautionary checking is actually required. Particularly, it must be shown that, for any complete metric space X, $\mathcal{F}(X)$ admits a complete metric space structure as well. Here we shall content ourselves with proving this, introducing some notations and a preparatory result.

We first recall that, for any metric space $\langle X, d\rangle$, the set of its closed subsets, denoted by $\mathcal{P}_{cl}(X)$, is naturally equipped with a metric, the so-called *Hausdorff distance* d_H which is defined as follows:

[23] The use of fixed-point theorems in connection with semantics for naive set theory in many-valued logics is detailed in [25].

For any $x \in X$ and $Y \in \mathcal{P}_{cl}(X)$, set

$$d_*(x, Y) := \inf_{y \in Y}\{d(x, y)\}$$

and then, for any $A, B \in \mathcal{P}_{cl}(X)$, define

$$d_H(A, B) := \max\Big\{\sup_{a \in A}\{d_*(a, B)\}, \sup_{b \in B}\{d_*(b, A)\}\Big\}.$$

In this way, it is proved that whenever $\langle X, d\rangle$ is complete, so is $\langle \mathcal{P}_{cl}(X), d_H\rangle$.[24] Now $\langle \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X), d_{\max}\rangle$ is also a complete metric space with

$$d_{\max}\big((A_1, B_1), (A_2, B_2)\big) := \max\big\{d_H(A_1, A_2), d_H(B_1, B_2)\big\}.$$

Thus, seeing that $\mathcal{F}(X) \subset \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X)$, it will be a complete metric space provided we show that $\mathcal{F}(X)$ is closed:

Proof. Let $(A_k, B_k) \underset{k\to\infty}{\longrightarrow} (A, B)$, with $(A_k, B_k) \in \mathcal{F}(X)$ for all $k \in \mathbb{N}$.

Suppose there exists $x_0 \in X$ with $x_0 \notin A \cup B$. Then we would have both $d_*(x_0, A) \geqslant \varepsilon$ and $d_*(x_0, B) \geqslant \varepsilon$, for some $\varepsilon > 0$. Now, since $A_k \underset{k\to\infty}{\longrightarrow} A$ and $B_k \underset{k\to\infty}{\longrightarrow} B$, one could also find $n \in \mathbb{N}$, such that both $d_H(A_n, A) < \varepsilon$ and $d_H(B_n, B) < \varepsilon$, and it would follow therefrom that $x_0 \notin A_n \cup B_n$, which is impossible. Whence $A \cup B = X$, and thus $(A, B) \in \mathcal{F}(X)$. $\square$

Remark 16. It should be noted that this result is very characteristic of the *paraconsistent* interpretation. For instance, the "pseudo-dual" operator $\mathcal{F}^*(\cdot)$ defined by $\mathcal{F}^*(X) := \{(A, B) \mid A, B$ closed in X and $A \cap B = \emptyset\}$ may no longer yield a complete metric space. Indeed, take $X := [-1, 1]$ with $(A_k, B_k) := ([-1, -\frac{1}{k}], [\frac{1}{k}, 1])$, for each $k \geqslant 1$. Then $(A_k, B_k) \underset{k\to\infty}{\longrightarrow} ([-1, 0], [0, 1]) \notin \mathcal{F}^*(X)$. Anyhow, it is apparent from Section 6.2 that a *paracomplete* set should rather be considered as an ordered pair of disjoint *open* subsets, not closed.

On the way, we conclude this section by proving a short preparatory result:

Lemma 17. *Let $f : X \to \mathcal{P}_{cl}(X)$ be continuous. Then* $\{(x, y) \in X^2 \mid x \in f(y)\} \in \mathcal{P}_{cl}(X^2)$.

Proof. Set $F = \{(x, y) \in X^2 \mid x \in f(y)\}$ and suppose $(x_k, y_k) \underset{k\to\infty}{\longrightarrow} (x, y)$ with $(x_k, y_k) \in F$ for all $k \in \mathbb{N}$. Then, since f is continuous, $f(y_k) \underset{k\to\infty}{\longrightarrow} f(y)$. Hence, for any $\varepsilon > 0$, there does exist n such that both $d(x_n, x) < \frac{\varepsilon}{2}$ and $d_H(f(y_n), f(y)) < \frac{\varepsilon}{2}$. As $x_n \in f(y_n)$, this implies $d_*(x_n, f(y)) < \frac{\varepsilon}{2}$ and then one can find $z \in f(y)$ with $d(x_n, z) < \frac{\varepsilon}{2}$. Thus we have $d(x, z) \leqslant d(x, x_n) + d(x_n, z) < \varepsilon$, showing that $x \in \overline{f(y)} = f(y)$, and so $(x, y) \in F$. $\square$

[24] See for instance Engelking's book: *General Topology*, Polish Scientific, Warsaw, 1977.

7. The model

It is remarkable that actually we do not even need to know the internal structure of M to define our model. Indeed, all we need is a compact metric space M together with an homeomorphism h (not necessarily an isometry) onto $\mathcal{F}(M)$:

$$M \overset{h}{\rightleftharpoons} \mathcal{F}(M).$$

We begin with specifying the interpretation of the primitive symbols in M:

$$[\cdot]_{\mathcal{M}} := h, \text{ i.e.: for any } a, b \text{ in } M, \quad \begin{cases} \mathtt{t} \in |a \in b|_{\mathcal{M}} & \text{:iff} \quad a \in (pr_1 \circ h)(b), \\ \mathtt{f} \in |a \in b|_{\mathcal{M}} & \text{:iff} \quad a \in (pr_2 \circ h)(b)([25]). \end{cases}$$

As h is 1–1, this defines a *strongly extensional* structure (in the sense of Section 2.1). In consequence, we may define the interpretation of '$=$' as follows:

$$\text{for any } a, b \text{ in } M, \quad \begin{cases} \mathtt{t} \in |a = b|_{\mathcal{M}} & \text{:iff} \quad a = b \text{ in } M, \\ \mathtt{f} \in |a = b|_{\mathcal{M}} & \text{:iff} \quad \mathtt{t} \in |\delta(a, b)|_{\mathcal{M}}. \end{cases}$$

Thus we get an extensional normal structure that is *perfect* (as defined in Section 4.2).

We now move on to the analogue of monotonicity in this context:

Definition 18. We define the *positive* and *negative extensions* of a $\mathcal{L}_M\langle\in, =\rangle$-formula $\varphi(\bar{u}_{(n)})$, with $\bar{u} = u_1, \dots, u_n$ and $n \geqslant 1$, respectively as follows:

$$\begin{cases} \langle\varphi(\bar{u})\rangle^+_{\mathcal{M}} := \{(\bar{a}) \in M^n \mid \mathtt{t} \in |\varphi(\bar{u}|\bar{a})|_{\mathcal{M}}\}, \\ \langle\varphi(\bar{u})\rangle^-_{\mathcal{M}} := \{(\bar{a}) \in M^n \mid \mathtt{f} \in |\varphi(\bar{u}|\bar{a})|_{\mathcal{M}}\}. \end{cases}$$

Then we say that such a formula $\varphi(\bar{u}_{(n)})$ is *continuous* if its positive and negative extensions thus defined are *closed* subsets of M^n.

It is worth observing that this actually does not depend on the choice of the free variables in φ, for it can be seen that $\varphi(\bar{u}_{(n)})$ is continuous iff $\varphi(\bar{v}_{(m)})$ is,[26] as well as if $\varphi(\bar{u}_{(k)}, \bar{v}_{(l)})$ is continuous and $(\bar{a}) \in M^l$, then so is $\varphi(\bar{v}|\bar{a})(\bar{u}_{(k)})$.

Following the rules defining the truth functions of the monotonic connectives and quantifiers, it is readily shown that:

$$\big\langle(\neg\varphi)(\bar{u})\big\rangle^+_{\mathcal{M}} = \big\langle\varphi(\bar{u})\big\rangle^-_{\mathcal{M}},$$
$$\big\langle(\neg\varphi)(\bar{u})\big\rangle^-_{\mathcal{M}} = \big\langle\varphi(\bar{u})\big\rangle^+_{\mathcal{M}};$$

$$\big\langle(\varphi \wedge \psi)(\bar{u})\big\rangle^+_{\mathcal{M}} = \big\langle\varphi(\bar{u})\big\rangle^+_{\mathcal{M}} \cap \big\langle\psi(\bar{u})\big\rangle^+_{\mathcal{M}},$$
$$\big\langle(\varphi \wedge \psi)(\bar{u})\big\rangle^-_{\mathcal{M}} = \big\langle\varphi(\bar{u})\big\rangle^-_{\mathcal{M}} \cup \big\langle\psi(\bar{u})\big\rangle^-_{\mathcal{M}};$$

[25] pr_1, pr_2 denote respectively the projections on the first and on the second component.

[26] We would remind the reader that by writing $\varphi(\bar{u}_{(n)})$ and $\varphi(\bar{v}_{(m)})$ the actual free variables of φ are supposed to be among $\bar{u} = u_1, \dots, u_n$ and $\bar{v} = v_1, \dots, v_m$.

$$\big\langle(\forall x\varphi)(\bar u)\big\rangle^+_{\mathcal M}=\bigcap_{a\in M}\big\langle\varphi(x|a)(\bar u)\big\rangle^+_{\mathcal M},$$
$$\big\langle(\forall x\varphi)(\bar u)\big\rangle^-_{\mathcal M}=pr_2\big\langle\varphi(x,\bar u)\big\rangle^-_{\mathcal M}.$$

In this latter we are rather considering $\langle\varphi(x,\bar u_{(n)})\rangle^-_{\mathcal M}$ as a subset of $M\times M^n$, so that pr_2 has to be understood as the projection on the second component M^n. Incidentally, we could have written

$$\big\langle(\forall x\varphi)(\bar u)\big\rangle^-_{\mathcal M}=\bigcup_{a\in M}\big\langle\varphi(x|a)(\bar u)\big\rangle^-_{\mathcal M}$$

but this would not have enabled us to prove the next lemma:

Lemma 19. *Any $\varphi(\bar u_{(n)})\in\mathcal L_M\langle\in,=;0,\neg,\wedge,\forall\rangle$ is continuous and $\langle\varphi(\bar u)\rangle^+_{\mathcal M}\cup\langle\varphi(\bar u)\rangle^-_{\mathcal M}=M^n$.*

Proof. The topological ingredients of the proof can be summarized by two facts:

(1) A space X is Hausdorff iff $\Delta_X:=\{(a,a)\mid a\in M\}$ is closed in X^2;
(2) If X_1,X_2 are compact and Hausdorff spaces, then the projection maps $pr_i:X_1\times X_2\to X_i$ $(i=1,2)$ are *closed*, i.e. for any closed subset $F\subseteq X_1\times X_2$, $pr_i(F)$ is closed in X_i $(i=1,2)$.

From the rules described above, it is clear that the continuity of a formula is preserved under the logical operators '$\neg$' and '$\wedge$'; because of the compactness of M, the preservation under '$\forall$' is now guaranteed by (2). Thus, it suffices to show that the atomic formulae are continuous. Using (1), (2) and the lemma in Section 6.4, this easily follows from the following observations:

$$\begin{array}{c|c}
\langle 0\rangle^+_{\mathcal M}=\emptyset & \langle 0\rangle^-_{\mathcal M}=M\\
\langle x\in y\rangle^+_{\mathcal M}=\big\{(x,y)\mid x\in(pr_1\circ h)(y)\big\} & \langle x\in y\rangle^-_{\mathcal M}=\big\{(x,y)\mid x\in(pr_2\circ h)(y)\big\}\\
\langle x\in x\rangle^+_{\mathcal M}=pr_1\big(\langle x\in y\rangle^+_{\mathcal M}\cap\Delta_M\big) & \langle x\in x\rangle^-_{\mathcal M}=pr_1\big(\langle x\in y\rangle^-_{\mathcal M}\cap\Delta_M\big)\\
\langle x=y\rangle^+_{\mathcal M}=\Delta_M & \langle x=y\rangle^-_{\mathcal M}=\big\langle\delta(x,y)\big\rangle^+_{\mathcal M}\\
\langle x=x\rangle^+_{\mathcal M}=M & \langle x=x\rangle^-_{\mathcal M}=pr_1\big(\langle x=y\rangle^-_{\mathcal M}\cap\Delta_M\big)
\end{array}$$

[Recalling that $\delta(x,y):\equiv\exists z((z\in x\wedge z\notin y)\vee(z\notin x\wedge z\in y))$ (see Section 4.2).] The second assertion of the lemma is easily proved by induction on the complexity of a formula as well, seeing that it does hold for atomic formulae. □

Now we are ready to prove the expected result:

Theorem 20. $\mathcal M\models$ *Comp*$[\mathcal L\langle\in,=;0,\neg,\wedge,\forall\rangle]$.

Proof. Let $\varphi(x)\in\mathcal L_M\langle\in,=;0,\neg,\wedge,\forall\rangle$. By the preceding lemma, $(\langle\varphi(x)\rangle^+_{\mathcal M},\langle\varphi(x)\rangle^-_{\mathcal M})$ is a covering pair of closed subsets of M. Therefore, by the surjectivity of h, there does exist b in M such that $[b]^+_{\mathcal M}=\langle\varphi(x)\rangle^+_{\mathcal M}$ and $[b]^-_{\mathcal M}=\langle\varphi(x)\rangle^-_{\mathcal M}$. □

Hereby the equality relation makes its entrance in formulae defining sets, within an extensional universe. According to Section 4.2, some non-monotonic functions should come along with it. In the model, this is certified by the next lemma that states that, beside the equality relation, other non-monotonic functions are allowed in formulae defining sets by means of restricted quantifications:

Lemma 21. *If φ is continuous, so are* '$\forall x(x \in y \to \varphi)$' *and* '$\forall x(x \notin y \to \varphi)$'.

Proof. Suppose that $\varphi(x, y, \bar{z})$ is continuous and let $\psi(y, \bar{z}) :\equiv \forall x(x \in y \to \varphi(x, y, \bar{z}))$. One the one hand, it must be shown that $F := \langle\psi(y, \bar{z})\rangle^+_{\mathcal{M}}$ is closed.

So, let $(y_k, \bar{z}_k) \underset{k\to\infty}{\longrightarrow} (b, \bar{c})$ with $(y_k, \bar{z}_k) \in F$, for any $k \in \mathbb{N}$.

Suppose $\mathtt{t} \in |a \in b|_{\mathcal{M}}$, for some $a \in M$. We have to show that $\mathtt{t} \in |\varphi(a, b, \bar{c})|_{\mathcal{M}}$.

As $(pr_1 \circ h)(\cdot) = [\cdot]^+_{\mathcal{M}}$ is continuous, we have $d_H([y_k]^+_{\mathcal{M}}), [b]^+_{\mathcal{M}}) \underset{k\to\infty}{\longrightarrow} 0$.

Now, as $a \in [b]^+_{\mathcal{M}}$, this implies $d_*(a, [y_k]^+_{\mathcal{M}}) \underset{k\to\infty}{\longrightarrow} 0$. Therefore, for any $n \in \mathbb{N}_0$, one can find $x_{k_n} \in [y_{k_n}]^+_{\mathcal{M}}$ such that $d(a, x_{k_n}) < 1/n$ (and such that $k_n < k_{n+1}$). Thus, as $\mathtt{t} \in |x_{k_n} \in y_{k_n}|_{\mathcal{M}}$ and $(y_{k_n}, \bar{z}_{k_n}) \in F$, we have $\mathtt{t} \in |\varphi(x_{k_n}, y_{k_n}, \bar{z}_{k_n})|_{\mathcal{M}}$, for any $n \in \mathbb{N}_0$. Now, since $(x_{k_n}, y_{k_n}, \bar{z}_{k_n}) \underset{k\to\infty}{\longrightarrow} (a, b, \bar{c})$ and φ is continuous, it follows that $\mathtt{t} \in |\varphi(a, b, \bar{c})|_{\mathcal{M}}$, as expected. Hence $\langle\psi(y, \bar{z})\rangle^+_{\mathcal{M}}$ is closed.

On the other hand, $\langle\psi(y, \bar{z})\rangle^-_{\mathcal{M}} = pr_2(\langle x \in y\rangle^+_{\mathcal{M}} \cap \langle\varphi(x, y, \bar{z})\rangle^-_{\mathcal{M}})$ is clearly closed.

We thus have proved that $\psi(y, \bar{z})$ is continuous.

The proof is similar for $\psi'(y, \bar{z}) :\equiv \forall x(x \notin y \to \varphi(x, y, \bar{z}))$. $\square$

We shall now review succinctly some properties of the model which show why such models can be considered as *natural models* for a paraconsistent set theory.

Let us first adopt the following abbreviation: '$x \sqsubseteq y$' $:\equiv$ '$\forall z(z \in x \Rightarrow z \in y)$'.

It is easily observed that $\mathcal{M} \models a \sqsubseteq b$ iff $[a]^+_{\mathcal{M}} \subseteq [b]^+_{\mathcal{M}}$ and $[b]^-_{\mathcal{M}} \subseteq [a]^-_{\mathcal{M}}$, and so:

$$\mathcal{M} \models a = b \quad \text{iff} \quad \mathcal{M} \models a \sqsubseteq b \wedge b \sqsubseteq a.$$

According to the last lemma, '$\sqsubseteq$' can be used in formulae defining sets, so that the existence of a *relevant* power-set operation, namely $\mathcal{P}(y) :=$ '$\{x \mid x \sqsubseteq y\}$', is actually derivable from the extended comprehension scheme the model fulfills.

From the algebraic point of view, let us mention that the universe of the model is a typical example of what is called a *paraconsistent boolean algebra* in [13]. Incidentally, the underlying order of this algebra is nothing but '$\sqsubseteq$' above. Note also that the subalgebra of *classical* sets is exactly (up to isomorphism) the (classical) boolean algebra of *clopen* subsets in M.

A very characteristic property of the model comes directly from its definition and can be expressed as follows:

Definition 22. For any $a, b \in M$, a is said to be *less contradictory* than b, denoted by $a \preceq b$, whenever $[a]^+_{\mathcal{M}} \subseteq [b]^+_{\mathcal{M}}$ and $[a]^-_{\mathcal{M}} \subseteq [b]^-_{\mathcal{M}}$. [Note that in the three-valued setting, this amounts to saying that $[a]_{\mathcal{M}} \leqslant_I [b]_{\mathcal{M}}$, namely $[a]_{\mathcal{M}}(x) \leqslant_I [b]_{\mathcal{M}}(x)$, for all $x \in M$, defining the information/knowledge ordering at the set level.]

So, with this terminology, we have:

$$\left|\begin{array}{l}\text{for each covering pair } (A, B) \text{ of subsets of } M\text{, there exists}\\ \text{a } \preceq\text{-minimal } x_0 \text{ in } M \text{ such that } A \subseteq [x_0]^+_{\mathcal{M}} \text{ and } B \subseteq [x_0]^-_{\mathcal{M}}.\end{array}\right.$$

Indeed, it is nothing but the (unique) x_0 such that $[x_0]_{\mathcal{M}} = (\overline{A}, \overline{B})$.

As an immediate consequence, any collection definable by a $\mathcal{L}$-formula $\varphi(x)$ is *optimally* approximated by a $\preceq$-minimal set x_0 defined by:

$$[x_0]_{\mathcal{M}} = \left(\overline{\langle\varphi(x)\rangle^+_{\mathcal{M}}}, \overline{\langle\varphi(x)\rangle^-_{\mathcal{M}}}\right).$$

The natural paraconsistent first-order theory to which all these properties give rise is axiomatized in [24]. There it was conjectured that, with a relevant formulation of an axiom of infinity, that theory could interpret ZF. Very recently, Esser [14] has proved the existence of models fulfilling such a relevant axiom of infinity. These are obtained from strong models of positive set theory called *hyperuniverses* and described in [18].

As a concluding remark, it should be mentioned that the dual structure $\mathcal{M}^*$ (as defined in Section 2.1) is *not* a solution to the corresponding *paracomplete* problem. Although $\mathcal{M}^*$ is a strongly extensional paracomplete model of *Comp*$[\mathcal{L}\langle\in; 0, \neg, \wedge, \forall\rangle]$, it can be shown that the "full" equality relation in formulae defining sets is still missing; actually, equality between classical sets only is allowed. This should be discussed elsewhere.

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